The Structure of Fully Prime Rings Under Ascending Chain Conditions (Logics, Algebras and Languages in Computer Science)

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0. Introduction

Definition *A ring R (possibly without identity) in which every ideal is prime (i.e. every proper ideal is a prime ideal and (hence) R is a prime ring) is called a fully prime ring.*

The purpose of my talk at the conference was to introduce the study of fully prime rings and present a few open problems to a wider range of audience. In this paper, a brief survey of results on the structure of fully prime rings with several examples is presented.
Our investigation for the structure of fully prime rings was initially motivated by a well-known fact that a commutative fully prime ring with identity is a field. We first published a series of papers on the subject in 1994 and 1996 (Blair-Tsutsui[1], Tsutsui [7]). Conditions similar to the fully prime condition have received attention in literature. Hirano [3] studied those rings in which every ideal is completely prime. Courter [2] studied those rings in which every ideal is semiprime, and Koh [6] studied those rings in which every right ideal is prime. More recently, Hirano-Tsutsui [5] studied rings in which every ideal is n-primary, and Hirano-Poon-Tsutsui [4] studied those rings in which every ideal is weakly prime. For many years after our initial publications, our focus has been to determine the structure of noncommutative right and left Noethrian fully prime ring.
Throughout this article, we assume a ring to be associative but not necessarily commutative. Due to the consideration that an ideal of a ring being a ring of its own, we do not assume the existence of a multiplicative identity on a ring unless otherwise so stated.

1. A few basic theorems on fully prime rings

Theorem 1 (Blair-Tsutsui [1]). The center of a fully prime ring is either a field or zero, and a ring R is fully prime if and only if every (two sided) ideal of R is idempotent and the set of ideals of R is totally ordered under inclusion.

In particular, as is well known, a commutative fully prime ring with identity is a field. Examples of a non-commutative fully prime ring include the ring of endomorphisms $Hom_D(V,V)$ of a vector space V over a division ring D. (This can easily be verified by the theorem above.) Denote the cardinality of a denumerable set by $\aleph_0$, and for any integer $n \geq 1$, let $\dim_D V = \aleph_{n-1}$. Then is fully prime with exactly k non-zero proper ideals $I_{\aleph_0} = \{f \in Hom_D(V,V) \mid \dim f(V) < \aleph_n\}$, $n = 0,2,\ldots,k-1$. If $\dim_D V = \aleph_{\omega_0}$ where ω_0 is the first limit ordinal, then $Hom_D(V,V)$ is a fully prime ring that has countably many ideals. Every right ideal, and hence every ideal of a regular ring is idempotent and for a regular self-injective rings *R*, an ideal *P* is prime if and only if *R*/*P* is totally ordered. Since there exists a regular self-injective ring *T* with a prime ideal *P* such that the set of ideals of *T*/*P* is not well-ordered, the set of ideals of a fully prime ring is not necessarily well-ordered.
Blair-Tsutsui[1] gives an example of a fully prime ring (with exactly one nonzero proper ideal) which is not primitive. A ring *R* is called fully isomorphic if *R*/*I* is isomorphic to *R* for every proper ideal *I* of *R*. Non-commutative fully isomorphic rings are fully prime, and there exists an example of a non-commutative fully isomorphic ring with infinitely many ideals, from which an example of a non-primitive fully prime ring with identity with infinitely many ideals may also be constructed.
Let *F* be a field, and *R* be the set of all infinite matrices over *F* that have the form

$$\begin{bmatrix} A & & & & & \\ & b & & 0 & & \\ & & c & & & \\ & & & b & & \\ & 0 & & & c & \\ & & & & & \ddots \end{bmatrix}$$

where *A* is an arbitrary $2n$ by $2n$ matrix and *b* and *c* are any elements of *F*. Then *R* is a prime ring all of whose ideals are idempotent that contains non-prime ideals.
Problem 1. Under what conditions, would a prime ring in which every ideal is idempotent be a fully prime ring?

Theorem 2 (Blair-Tsutsui [1]). Let R be a fully prime ring. Then every ideal of R is fully prime when it is considered as a ring without identity. Every ideal of an ideal of R is an ideal of R. Further, a proper ideal of R cannot be a ring with identity.

If P is a proper ideal of a fully prime ring R with identity and F is a subfield of the center of R, then $S_P = F + P$ is a fully prime ring whose maximal ideal P is also a maximal right and left ideal. Further, proper ideals of S are precisely those ideals of R that are contained in P.

Theorem 3 (Blair-Tsutsui [1]). $S_P = F + P$ is right primitive if and only if R is right primitive. S_P is semiprimitive if and only if R is semiprimitive.

2. Right Noetherian Fully Prime Rings

In this section, we assume that a ring has a multiplicative identity.

Theorem 4 (Blair-Tsutsui [1]). A fully prime, right fully bounded right Noetherian ring is simple Artinian.

More generally, a prime right fully bounded right Noetherian ring, all of whose ideals are idempotent, is a simple Artinian ring (Tsutsui [7]).
As shown in Blair-Tsutsui [1], a fully prime ring is, in general, not semiprimitive. By Nakayama's lemma, it is evident, however, that a right Noetherian fully prime ring is semiprimitive. Further, an induction argument on the Krull dimension yields the following result.

Theorem 5 (Tsutsui [8]). A fully prime ring with right Krull dimension is semiprimitive.

Let $A_1(k)$ be the first Weyl algebra over a field of characteristic 0. Then $S = k + xA_1(k)$ is a right and left Notherian fully prime ring with exactly one nonzero proper ideal $xA_1(k)$. Thus, Noetherian fully prime ring is not necessarily a simple ring.

Theorem 6 (Tsutsui [8]). Let $S = k + xA_1(k)$ where the characteristic of k is zero. Then the idealizer of any maximal right ideal M of S different from $xA_1(k)$ is not a fully prime ring.

Whether there exists a fully prime right Noetherian ring with more than one nonzero proper ideals remains a tantalizing open question. Let $R = k + xA_1(k) \otimes_k (k + xA_1(k))$. Then R has exactly two nonzero proper ideals $xA_1(k) \otimes_k xA_1(k)$ *and* $xA_1(k) \otimes_k (k + xA_1(k))$ and R is a fully prime ring.

Problem 2. Is $R = k + xA_1(k) \otimes_k (k + xA_1(k))$ a right Noetherian ring?

Theorem 7. (Tsutsui [8]). A fully prime ring with identity that has a right Krull dimension and a minimal nonzero ideal is a right primitive ring.

Let R be a ring with identity and $\mathfrak{M}_R$ be the class of all right R-modules. The class of modules that are closed under taking factors, sum, extensions, and submodules is called a hereditary torsion class. A hereditary torsion class is called a TTF-class if it is also closed under taking direct products. It is well-known that $\mathfrak{I}$ is a TTF-class if and only if $\mathfrak{I} = \{M \in \mathfrak{M}_R \mid MI = 0\}$ for some idempotent ideal I of R. Since every ideal P of a fully prime ring R is idempotent and the set of ideals is linearly ordered, the TTF-classes $K_P = \{M \in \mathfrak{M}_R \mid MP = 0\}$ of a fully prime ring R are linearly ordered. Let $\mathfrak{A} = \bigcup_{0 \neq P \triangleleft R} K_P$ where R is a fully prime ring. It is evident that $\mathfrak{A}$ is a hereditary torsion class. Let $\bar{\mathfrak{A}}$ be the smallest TTF-class containing $\mathfrak{A}$. Then since $K_P \subseteq \bar{\mathfrak{A}}$ for every nonzero ideal P, $\bar{\mathfrak{A}} = K_{\bigcap_{0 \neq P \triangleleft R} P}$. Considering a module representation of R in $\bar{\mathfrak{A}}$, we are strongly hopeful to prove, by contradiction, that a fully prime ring is primitive if and only if it is semiprimitive. This result, if affirmative, will in particular imply that a fully prime ring is primitive if it is von Neumann regular. It would also imply that a fully prime right Noetherian ring is primitive with finitely many ideals.

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